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► To cite this version:

Achour Mostefaoui, Michel Raynal, Julien Stainer. Relations Linking Failure Detectors Associated with k-Set Agreement in Message-Passing Systems. [Research Report] PI-1973, 2011, pp.13. inria-00583301

HAL Id: inria-00583301

<https://hal.inria.fr/inria-00583301>

Submitted on 5 Apr 2011

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Relations Linking Failure Detectors Associated with k -Set Agreement in Message-Passing Systems

Achour Mostéfaoui* Michel Raynal** Julien Stainer***

Abstract: The k -set agreement problem is a coordination problem where each process is assumed to propose a value and each process that does not crash has to decide a value such that each decided value is a proposed value and at most k different values are decided. While it can always be solved in synchronous systems, k -set agreement has no solution in asynchronous send/receive message-passing systems where up to $t \geq k$ processes may crash.

A failure detector is a distributed oracle that provides processes with additional information related to failed processes and can consequently be used to enrich the computability power of asynchronous send/receive message-passing systems. Several failure detectors have been proposed to circumvent the impossibility of k -set agreement in pure asynchronous send/receive message-passing systems. Considering three of them (namely, the generalized quorum failure detector Σ_k , the generalized loneliness failure detector \mathcal{L}_k and the generalized eventual leader failure detector Ω_k) the paper investigates their computability power and the relations that link them. It has three main contributions: (a) it shows that the failure detector Ω_k and the eventual version of \mathcal{L}_k have the same computational power; (b) it shows that \mathcal{L}_k is realistic if and only if $k \geq n/2$; and (c) it gives an exact characterization of the difference between \mathcal{L}_k (that is too strong for k -set agreement) and Σ_k (that is too weak for k -set agreement).

Key-words: Asynchronous message-passing system, Distributed computability, Equivalence, Eventual leader, Failure detector, Fault-tolerance, Impossibility, Quorum, Realistic failure detector, Reduction, k -Set agreement, Theory.

Comprendre les détecteurs de fautes associés au problème de k -accord

Résumé : Ce rapport reprend les différents détecteurs de fautes introduits pour renforcer les systèmes asynchrones et rendre le problème de k -accord décidable. En effet, plusieurs détecteurs de fautes ont été introduits mais certains sont trop forts et d'autres trop faibles. Ce rapport a pour but de déterminer l'espace qui sépare les premiers des seconds afin d'essayer de se rapprocher du détecteur minimal.

Mots clés : détecteur de fautes, k -accord, leader, tolérance aux fautes, oracle, réduction algorithmique, système asynchrone.

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1 Introduction

On failure detectors Let us observe that in asynchronous systems where the only means for processes to communicate is using send/receive message-passing, no process is able to know if another process has crashed or is only very slow. The concept of a failure detector originates from this simple observation. A failure detector is a device (distributed oracle) whose aim is to enrich a distributed system by providing alive processes with information on failed processes [6]. Several classes of failure detectors can be defined according to the type of information on failures they provide to processes (see [16] for an introduction to failure detectors).

The k -set agreement problem This problem, that has been introduced by S. Chaudhuri [8], is a coordination problem that generalizes the consensus problem. It can be defined as follows [8, 15]. Each process proposes a value and every non-faulty process has to decide a value (termination) in such a way that any decided value is a proposed value (validity) and no more than k different values are decided (agreement). The problem parameter k defines the coordination degree: $k = 1$ corresponds to its most constrained instance (consensus) while $k = n - 1$ corresponds to its weakest non-trivial instance (called set agreement).

Let t be the model parameter that denotes the upper bound on the number of processes that may crash in a run, $1 \leq t < n$. If $t < k$, k -set agreement can be trivially solved in both synchronous and asynchronous systems: k predetermined processes broadcast the values they propose and a process decides the first proposed value it receives. Hence, the interesting setting is when $k \geq t$, i.e., when the number of values that can be decided is smaller or equal to the maximal number of processes that may crash in a run.

Algorithms that solve the k -set agreement problem in synchronous message-passing systems when $k \leq t$ are presented in [1, 13, 18]. These algorithms are based on a sequence of synchronous communication rounds. It is shown in the three books previously referenced that $\lfloor \frac{t}{k} \rfloor + 1$ rounds are necessary and sufficient to solve k -set agreement. (This lower bound is still valid in more severe failure models such as general omission failures [18].)

For crash-prone asynchronous systems (be the communication medium a read/write shared memory or a send/receive message-passing network) the situation is different, namely, the k -set agreement problem has no solution when $t \geq k$ [5, 12, 20].

The cases $k = 1$ and $k = n - 1$ in message-passing systems When $k = 1$, as already indicated k -set agreement boils down to consensus, and it is known that the failure detector denoted Ω is the weakest to solve consensus in asynchronous message-passing systems where $t < n/2$ [7]. Ω ensures that there is an unknown but finite time after which all the processes have the same non-faulty leader (before that time, there is an anarchy period during which each process can have an arbitrarily changing leader). This lower bound result is generalized in [10] where the failure detector Σ is introduced and it is shown that the pair $\langle \Sigma, \Omega \rangle$ is the weakest failure detector to solve consensus in message-passing systems when $t < n$. This means that Σ is the minimal additional power (as far as information on failures is concerned) required to overcome the barrier $t < n/2$ and attain $t \leq n - 1$. Actually, the power provided by Σ is the minimal one required to implement a shared register in a message-passing system [4, 10]. Σ provides each process with a quorum (set of process identities) such that the values of any two quorums (each taken at any time) intersect, and there is a finite time after which any quorum includes only correct processes. Fundamentally, Σ prevents partitioning. A failure detector $\langle \Sigma, \Omega \rangle$ outputs a pair of values, one for Σ and one for Ω .

The *Loneliness* failure detector (denoted \mathcal{L}) has been proposed in [11] where it is proved that it is the weakest failure detector for solving $(n - 1)$ -set agreement in the asynchronous message-passing model. Such a failure detector provides each process p with a boolean (that p can only read) such that the boolean of at least one process remains always false and, if all but one process crash, the boolean of the remaining process becomes and remains true forever. It is important to notice that, albeit surprisingly, the weakest failure detector for $(n - 1)$ -set agreement is not the same in the read/write shared memory model (where it is $\overline{\Omega}_{n-1}$) and the send/receive message-passing model (where it is \mathcal{L}).

Unfortunately, the weakest failure detector for k -set agreement when $1 < k < n - 1$ in message-passing asynchronous crash-prone systems is not yet known. The interested reader will find an introductory survey on failure detectors suited to k -set agreement in [19].

Content of the paper Among the failure detectors for k -set agreement, $1 \leq k \leq n - 1$, that have been proposed in the past few years, this paper investigates the relations on three of them, namely the ones denoted \mathcal{L}_k , Ω_k and Σ_k (their precise definitions are given in Section 3). The failure detector Ω_k , introduced in [14], is a generalization of Ω (Ω_1 is Ω).

The failure detector \mathcal{L}_k , introduced in [2], is a generalization of \mathcal{L} . It allows the k -set agreement problem to be solved in message-passing despite asynchrony and any number of process crashes. Unfortunately, \mathcal{L}_k has been proved to be (a little bit) too strong to solve k -set agreement. Hence, the question “How much stronger is it?”

The failure detector Σ_k , introduced in [3], is a generalization of Σ . It is shown in [3] that Σ_k is necessary (but unfortunately not sufficient) to solve k -set agreement. Hence, the question “How much weaker is it?” It is also shown in [3] that Σ_{n-1} and \mathcal{L}_{n-1} are equivalent (they provide processes with the same computational power).

Answering the two previous questions seems difficult as it would provide us with key elements to obtain the weakest failure detector for asynchronous message-passing k -set agreement. Hence, we consider a more modest question in this paper whose answer will help us better understand and pave the way to the discovery of the weakest failure detector for message-passing k -set agreement. The question is “Which is the property that has to be added to Σ_k in order to obtain exactly \mathcal{L}_k ?” To be more explicit let X_k be this property. Answering this question means solving the equation $\langle \Sigma_k, X_k \rangle \simeq \mathcal{L}_k$ where X_k is the unknown and \simeq means “have the same computational power”. This paper has three contributions.

- It first focuses on the implementability of \mathcal{L}_k in a synchronous system. Let us remember that a failure detector is *realistic* if it can be implemented in a synchronous system [9]. The paper shows that $k \geq n/2$ is a necessary and sufficient requirement for \mathcal{L}_k to be realistic.
- It then answers the previous question by giving X_k .
- It finally shows that the “eventual” version of \mathcal{L}_k , which we denote $\Diamond \mathcal{L}_k$, is nothing else than Ω_k . (“Eventual” means that the property of \mathcal{L}_k are required to be satisfied only after some finite time).

Roadmap The paper is made up of 8 sections. Section 2 presents the base computation model and the k -set agreement problem. Section 3 introduces the failure detectors in which we are interested, i.e., Σ_k , \mathcal{L}_k and Ω_k . Section 4 presents an \mathcal{L}_k -based k -set agreement algorithm due to [2]. The next three sections are the contributions of the paper: Section 5 is on the realism of \mathcal{L}_k , Section 6 solves the equation $\langle \Sigma_k, X_k \rangle \simeq \mathcal{L}_k$, while Section 7 shows that $\Diamond \mathcal{L}_k$ and Ω_k are equivalent. Finally, Section 8 concludes the paper.

2 Base computation model and k -set agreement

2.1 Computation model

Process model The system consists of a set of n sequential processes denoted p_1, \dots, p_n . $\Pi = \{1, \dots, n\}$ is the set of process identities. Each process executes a sequence of (internal or communication) atomic steps. A process executes its code until it possibly crashes (if it ever crashes). After it has crashed, a process executes no more step. A process that crashes in a run is said *faulty* in that run, otherwise it is *correct*. Given a run, \mathcal{C} and \mathcal{F} denote the set of processes that are correct and the set of processes that are faulty, respectively. Up to $t = n - 1$ processes may crash in a run, hence, $1 \leq |\mathcal{C}| \leq n$.

Communication model The processes communicate by executing atomic communication steps which are the sending or the reception of a message. Every pair of processes is connected by a bidirectional channel. The channels are failure-free (no creation, alteration, duplication or loss of messages) and asynchronous (albeit the time taken by a message to travel from its sender to its receiver is finite, there is no bound on transfer delays). The notation “broadcast $\text{MSG_TYPE}(m)$ ” is used as a (non-atomic) shortcut for “**for each** $j \in \Pi$ **do** send $\text{MSG_TYPE}(m)$ to p_j **end for**”.

Underlying time model The underlying *time model* is the set \mathbb{N} of natural integers. As we are in an asynchronous system, this time notion is not accessible to processes (hence, the model is sometimes called time-free model). It can only be used from an external observer point of view to state or prove properties. Time instants are denoted τ, τ' , etc.

Notation The previous asynchronous crash-prone message-passing system model is denoted $\mathcal{AMP}[\emptyset]$. \mathcal{AMP} stands for “Asynchronous Message-Passing”; \emptyset means the “base” system (not enriched with a failure detector).

2.2 The k -set agreement problem

As already indicated, the k -set agreement problem has been introduced by Soma Chaudhuri [8]. It generalizes the consensus problem (that corresponds to $k = 1$). It is defined as follows. Each process proposes a value and has to decide a value in such a way that the following properties are satisfied:

- Termination. Every correct process decides a value.
- Validity. A decided value is a proposed value.
- Agreement. At most k different values are decided.

3 The failure detectors Ω_k , Σ_k and \mathcal{L}_k

This section presents the three failure detectors we are interested in. (People interested in the underlying assumptions and algorithms to implement failure detectors in asynchronous systems can consult Chapter 7 of [17].)

The system model $\mathcal{AMP}[\emptyset]$ enriched with any of these failure detectors \mathcal{A} is denoted $\mathcal{AMP}[\mathcal{A}]$. A failure detector provides each alive process p_i with a read-only local variable, say xxx_i . The value of xxx_i at time τ is denoted xxx_i^τ .

3.1 The eventual leadership failure detector Ω_k

The failure detector Ω_k has been introduced in [14]. Its local output at p_i is a set denoted $leaders_i$. Ω_k is defined by the two following properties.

- Validity. $\forall i, \forall \tau: (leaders_i^\tau \subset \Pi) \wedge (|leaders_i^\tau| = k)$.
- Eventual leadership. $\exists LD, \exists \tau: (LD \cap \mathcal{C} \neq \emptyset) \wedge (\forall \tau' \geq \tau, \forall i \in \mathcal{C}: leaders_i^{\tau'} = LD)$.

Validity means that the values of $leaders_i$ are k process identities. Eventual leadership states that, after some unknown but finite time, all correct processes have the same set of leaders and at least one of these leaders is a correct process. Before all processes are provided with the same set of leaders, there is possibly an unknown but finite anarchy period during which the sets $leaders_i$ have arbitrary values.

The failure detector from which Ω_k originates is Ω (which is the same as Ω_1). As already noticed, Ω has been introduced in [7] where it is shown to be the weakest failure detector to solve consensus in message-passing systems with a majority of correct processes. It has later been shown in [10] that $\langle \Sigma, \Omega \rangle$ is the weakest failure detector to solve consensus for $t = n - 1$.

3.2 The quorum failure detector Σ_k

The failure detector Σ_k has been introduced in [3]. It is a generalization of the failure detector Σ (called quorum failure detector) introduced in [10] where it is shown to be the weakest failure detector to implement a register in $\mathcal{AMP}[\emptyset]$ (Σ_1 is Σ).

The local output at p_i of Σ_k is a set qr_i (called quorum) that satisfies the following properties.

- Liveness. $\exists \tau: \forall i \in \mathcal{C}, \forall \tau' \geq \tau: qr_i^{\tau'} \subseteq \mathcal{C}$.
- Intersection. Let $\mathcal{I} = \{id_x\}_{1 \leq x \leq k+1} \subset \Pi$ be a multiset of $k + 1$ process identities. Let $\mathcal{T} = \{\tau_x\}_{1 \leq x \leq k+1}$ be a multiset of $k + 1$ time instants. $\forall \mathcal{I}, \forall \mathcal{T}: \exists i, j \in [1..k + 1]: (i \neq j) \wedge (qr_{id_i}^{\tau_i} \cap qr_{id_j}^{\tau_j} \neq \emptyset)$.

The liveness property states that the quorum of a correct process eventually includes only correct processes. The intersection property states that any set of $k + 1$ quorums, whose values are taken at any times, contains two intersecting quorums. This means that the intersection property prevents processes to partition in more than k subsets. It is shown in [3] that Σ_k is necessary when one wants to solve k -set agreement in $\mathcal{AMP}[\emptyset]$.

3.3 The loneliness failure detector \mathcal{L}_k

The \mathcal{L}_k family introduced in [2] is a generalization of the failure detector \mathcal{L} proposed in [11] where it is shown that \mathcal{L} is the weakest failure detector for $(n - 1)$ -set agreement in asynchronous message-passing systems. \mathcal{L}_{n-1} is \mathcal{L} .

The local output at p_i of \mathcal{L}_k is a boolean variable $alone_i$ that satisfies the following two properties (after a process p_i has crashed, we have $alone_i = false$ by definition).

- **Stability.** $\exists K \subset \Pi : (|K| = n - k) \wedge (\forall i \in K, \forall \tau : alone_i^\tau = false)$.
- **Loneliness.** $(|\mathcal{C}| \leq n - k) \Rightarrow (\exists i \in \mathcal{C}, \exists \tau : \forall \tau' \geq \tau : alone_i^{\tau'} = true)$.

Stability states that the boolean variables of at most k processes can take the value *true*, while the loneliness property states that, if at least k processes crash, then there is a finite time after which there is a correct process p_i whose boolean variable $alone_i$ remains forever equal to *true*. An algorithm solving the k -set agreement problem in $\mathcal{AMP}[\mathcal{L}_k]$ due to [2] is described in the next section. It is also shown in [2] that, for $1 < k < n - 1$, \mathcal{L}_k is not the weakest failure detector for k -set agreement in $\mathcal{AMP}[\emptyset]$.

4 A k -set agreement algorithm for $\mathcal{AMP}[\mathcal{L}_k]$

Algorithm 1 is an \mathcal{L}_k -based k -set agreement algorithm due to Biely, Robinson and Schmid [2]. This algorithm is based on a sequence of asynchronous rounds (r_i denotes the current round number of process p_i). The local variable est_i is p_i 's current estimate of its decision value; it is initialized to the value v_i proposed by p_i . The execution of the statement $\text{return}(v)$ returns the value v and terminates the invocation of $\text{set_agreement}_k()$.

```

when operation  $\text{set\_agreement}_k(v_i)$  is invoked by  $p_i$ :
(01)  $est_i \leftarrow v_i; r_i \leftarrow 1;$ 
(02) repeat forever
(03)   for each  $j \neq i$  do send  $\text{EST}(r_i, est_i)$  to  $p_j$  end for;
(04)   wait until  $((n - k)$  messages  $\text{EST}(r_i, -)$  have been received  $);$ 
(05)    $est_i \leftarrow \min(est_i, \text{the } est_j \text{ received at the previous line});$ 
(06)   if  $(r_i = k + 1)$ 
(07)     then for each  $j \neq i$  do send  $\text{DEC}(est_i)$  to  $p_j$  end for;
(08)     return  $(est_i)$ 
(09)   else  $r_i \leftarrow r_i + 1$ 
(10)   end if
(11) end repeat.

when  $(alone_i \vee \text{DEC}(v))$  is received):
(12) if  $(\text{DEC}(v))$  received then  $est_i \leftarrow v$  end if;
(13) for each  $j \neq i$  do send  $\text{DEC}(est_i)$  to  $p_j$  end for;
(14) return  $(est_i)$ .

```

Algorithm 1: k -Set agreement in $\mathcal{AMP}[\mathcal{L}_k]$ algorithm (code for p_i) [2]

Algorithm description In each round, each non-crashed process p_i first broadcasts a message $\text{EST}(r_i, est_i)$ (line 03) to inform the other processes of its current estimate est_i and waits until it has received $(n - k)$ estimate messages associated with its current round r_i (line 04). When it has received these estimates, it computes the smallest of them including its own estimate (line 05). Then if $r_i < k + 1$, p_i proceeds to the next asynchronous round (line 09). If $r_i = k + 1$, it broadcasts $\text{DEC}(est_i)$ to inform the other processes on the value it is about to decide (line 07) and then decides it (line 08).

When considering lines 01-11 only, let us observe that p_i can block forever at line 04 if more than k processes have crashed. Such a permanent blocking is prevented by the combined use of the failure detector \mathcal{L}_k and $\text{DEC}()$ messages (which ensures that, as soon as a process decides, all correct processes eventually decide). Observing that the boolean $alone_i$ of at least one correct process p_i becomes true when the number of alive processes becomes smaller than or equal to $n - k$, we can conclude that this correct process p_i unblocks the situation when this condition becomes satisfied.

On the power of \mathcal{L}_k It is shown in [2] that, for $n > 2$ and $k \geq 2$, \mathcal{L}_k is either weaker than or not comparable to Σ . As (a) consensus can be solved in both system models $\mathcal{AMP}[\mathcal{L}_1]$ and $\mathcal{AMP}[\Sigma, \Omega]$, and (b) $\mathcal{AMP}[\Sigma, \Omega]$ is the weakest failure detector-based model in which consensus can be solved, it follows that $\mathcal{AMP}[\mathcal{L}_1]$ is not the weakest failure detector-based model in which consensus can be solved. It also follows that, while $\mathcal{L}_{n-1} = \mathcal{L}$ is the weakest failure detector for $(n-1)$ -set agreement [11], \mathcal{L}_k , $1 \leq k < n-1$, is not the weakest failure detector for k -set agreement. But, as shown in the following theorem, \mathcal{L}_k seems to be not too much stronger than what is necessary.

Theorem 1 [2] *Let $2 \leq k \leq n-1$. k -Set agreement can be solved in $\mathcal{AMP}[\mathcal{L}_k]$. Moreover, $(k-1)$ -set agreement cannot be solved in $\mathcal{AMP}[\mathcal{L}_k]$. (Proof in [2].)*

5 On the implementability of \mathcal{L}_k in a synchronous system

This section addresses the realism of \mathcal{L}_k , i.e., its implementability in a synchronous distributed system.

Synchronous distributed system Synchrony is an abstraction that encapsulates (and hides) specific timing assumptions. A synchronous system provides the processes with a global clock r called *round number* which entails the progress of the computation. The processes progress simultaneously from round to round. During a round, a process sends a message to all processes, receives messages from other processes and executes local computation. The fundamental property associated with a synchronous system is that a message sent by a process during a round r is received by the other processes during the very same round r . (More information on synchronous systems can be found in the book [18] that is entirely devoted to such systems.)

Let \mathcal{SMP} denotes a *Synchronous Message-Passing* system made up of n processes, in which up to $t = n-1$ processes may crash.

Building \mathcal{L}_k in \mathcal{SMP} when $k \geq n/2$ Algorithm 2 presents the code of such a construction for a process p_i . Initially, $alone_i$ is false. Then, during each round r , p_i broadcasts a message $\text{ALIVE}(i)$ to inform the other processes that it was alive at the beginning of round r (line 04). Then, p_i receives round r messages and updates rec_ids_i accordingly (line 05). Finally, if rec_ids_i contains at most $n-k$ process identities, p_i sets $alone_i$ to *true* (line 06).

```

(01) init  $alone_i \leftarrow \text{false}$ ;
(02) when  $r = 1, 2, \dots$  do
(03)   begin synchronous round
(04)     broadcast  $\text{ALIVE}(i)$ ;
(05)      $rec\_ids_i \leftarrow \{j \text{ such that } \text{ALIVE}(j) \text{ received during the current round}\}$ ;
(06)     if  $(|rec\_ids_i| \leq n-k)$  then  $alone_i \leftarrow \text{true}$  end if
(07)   end synchronous round.

```

Algorithm 2: Building \mathcal{L}_k in \mathcal{SMP} when $k \geq n/2$ (code for p_i)

Lemma 1 *Let $k \geq n/2$. Algorithm 2 builds a failure detector \mathcal{L}_k in \mathcal{SMP} .*

Proof Proof of the stability property. Let r be the first round during which a process p_i sets $alone_i$ to *true*. This means that at least k processes have crashed before sending their round r message to p_i . Consequently, at most $n-k$ processes will ever set their boolean $alone_i$ to *true*. As $k \geq n/2 \Rightarrow n-k \leq k$, it follows that at least k processes will never set their boolean variables to *true* which proves the property.

Proof of the loneliness property. If k or more processes crash during a run, for each correct process p_i , there is a round r such that p_i receives at most $n-k$ messages at every round $r' \geq r$, from which it follows that, from round r , $alone_i$ remains forever equal to *true*. \square Lemma 1

Lemma 2 *It is not possible to build \mathcal{L}_k in \mathcal{SMP} when $k < n/2$.*

Proof Assuming by contradiction that there is a synchronous algorithm \mathcal{A} that builds \mathcal{L}_k (with $k < n/2$) in \mathcal{SMP} , let us consider a run r_{k+1} of \mathcal{A} in which the processes p_1, p_2, \dots, p_k have initially crashed. It follows from the loneliness property (guaranteed by \mathcal{A}) that there is a process (say p_{k+1}) whose boolean $alone_{k+1}$ becomes eventually *true* and remains afterward forever equal to *true*.

Let us now consider a second run r_{k+2} of \mathcal{A} that is identical to r_{k+1} until $alone_{k+1}$ becomes and remains forever equal to *true* and such that, after that round, p_{k+1} crashes. As previously, it follows from the loneliness property (guaranteed by \mathcal{A}) that there is a process (say p_{k+2}) whose boolean $alone_{k+2}$ becomes and remains forever equal to *true*. It is possible to continue to design similar runs r_{k+3} , etc., until r_n (in each r_x , $k+1 \leq x \leq n$, $x-1$ processes crash).

Hence, in the run r_n , $n-k$ processes p_j have set their boolean variable $alone_j$ to *true*. As $k < n/2 \Rightarrow n-k > k$, this contradicts the fact that the algorithm \mathcal{A} guarantees the stability property of \mathcal{L}_k , namely, at most k processes never set their boolean variable to *true*. A contradiction from which the lemma follows. $\square_{\text{Lemma 2}}$

The next theorem follows from Lemma 1 and Lemma 2.

Theorem 2 $k \geq n/2$ is a necessary and sufficient condition for \mathcal{L}_k to be realistic.

Let us notice that the failures detectors \mathcal{P} (perfect failure detector) [6], Σ , Σ_k , Ω , Ω_k are realistic.

6 Relating \mathcal{L}_k and Σ_k

This section determines the property that has to be added to Σ_k (which is necessary to solve k -set agreement) in order to obtain exactly \mathcal{L}_k . Let X_k be this property. Hence, this section solves the equation $\langle \Sigma_k, X_k \rangle \simeq \mathcal{L}_k$ where X_k is the unknown and $A \simeq B$ means “ $\mathcal{AMP}[A]$ and $\mathcal{AMP}[B]$ have the same computational power”.

6.1 The property X_k

X_k is designed to work with Σ_k . It is defined by the following property where qr_i is the output of Σ_k at p_i .

- Loneliness. $(|\mathcal{C}| \leq n-k) \Rightarrow (\exists i \in \mathcal{C}, \exists \tau : qr_i^\tau = \{i\})$.

Hence, X_k requires that, when at least k processes are faulty, there a time instant at which the quorum of a correct process contains only itself.

The next theorem follows directly from the lemmas 3 and 4 that are proved in the next two sections.

Theorem 3 $\mathcal{AMP}[\mathcal{L}_k]$ and $\mathcal{AMP}[\langle \Sigma_k, X_k \rangle]$ have the same computational power.

6.2 Building \mathcal{L}_k in $\mathcal{AMP}[\langle \Sigma_k, X_k \rangle]$

Algorithm 3 presents a very simple algorithm that builds \mathcal{L}_k in $\mathcal{AMP}[\langle \Sigma_k, X_k \rangle]$. The output $alone_i$ of each process p_i is initialized to *false* and is set to *true* if the predicate $qr_i = \{i\}$ becomes satisfied at least once.

```

init  $alone_i \leftarrow \text{false};$ 
when  $qr_i = \{i\}$ :  $alone_i \leftarrow \text{true}.$ 

```

Algorithm 3: Building \mathcal{L}_k in $\mathcal{AMP}[\langle \Sigma_k, X_k \rangle]$

Lemma 3 Algorithm 3 builds \mathcal{L}_k in $\mathcal{AMP}[\langle \Sigma_k, X_k \rangle]$.

Proof The intersection property of Σ_k guarantees that at most k processes will execute the body of the **when** statement. It follows that at most k processes p_i will ever set their boolean to *true* which (combined with the initialization of the variables $alone_i$) proves the stability property of \mathcal{L}_k .

The loneliness property of X_k implies that if k or more processes crash, there a time instant at which the predicate $qr_i = \{i\}$ will be satisfied at a correct process p_i . Hence, this process p_i will execute the body of the **when** statement and we will then have $alone_i = \text{true}$ forever, which proves the loneliness property of \mathcal{L}_k . $\square_{\text{Lemma 3}}$

Remark Let us notice that the previous proof does not use the liveness property of Σ_k .

6.3 Building $\langle \Sigma_k, X_k \rangle$ in $\mathcal{AMP}[\mathcal{L}_k]$

Underlying principle and description of the algorithm Algorithm 4 relies on the observation of the predicate $qr_i = \{i\}$ that it makes stable (once satisfied, it remains forever satisfied).

The variable qr_i of each process is initialized to Π in order not to compromise the intersection property. Then, if $alone_i$ becomes true, p_i sets qr_i to $\{i\}$ (line 03) and, from then on, qr_i will keep that value forever and p_i repeatedly informs of it the other processes by broadcasting a message $\text{ALONE}(i)$.

When it receives $\text{ALONE}(j)$ from another process (line 04), p_i learns that qr_j is keeping forever the value $\{j\}$. If $qr_i \neq \{i\}$, p_i updates accordingly qr_i to $\{i, j\}$ in order to preserve the intersection property of Σ_k (line 05).

Independently of its other statements, p_i repeatedly informs the other processes that it is alive (task $T1$ where messages $\text{ALIVE}(i)$ are broadcast forever). This triggers coordination among the processes even if no boolean $alone_i$ becomes equal to *true*. Its aim is to ensure the liveness property of Σ_k .

The task $T2$ is associated with the processing of the $\text{ALIVE}(j)$ received by p_i . When it has received such messages from $(n - k)$ distinct processes, if qr_i has not stabilized on $\{i\}$, p_i resets qr_i to the set containing i and the processes from which alive messages have been received.

```

(01) init:  $qr_i \leftarrow \Pi$ ; start  $T1, T2$ .

(02) when  $alone_i$  becomes equal to true:
(03)    $qr_i \leftarrow \{i\}$ ; repeat forever broadcast  $\text{ALONE}(i)$  end repeat.

(04) when  $\text{ALONE}(j)$  with  $j \neq i$  is received:
(05)   if  $(qr_i \neq \{i\})$  then  $qr_i \leftarrow \{i, j\}$  end if.

(06) task  $T1$ : repeat forever broadcast  $\text{ALIVE}(i)$  end repeat.

(07) task  $T2$ :
(08)   repeat forever
(09)     wait until (new  $\text{ALIVE}(j)$  messages with  $j \neq i$  received from  $n - k$  processes);
(10)     let  $proc_i$  = the set of  $n - k$  processes from which messages have been received;
(11)     if  $(qr_i \neq \{i\})$  then  $qr_i \leftarrow \{i\} \cup proc_i$  end if
(12)   end repeat.

```

Algorithm 4: Building $\langle \Sigma_k, X_k \rangle$ in $\mathcal{AMP}[\mathcal{L}_k]$

Lemma 4 Algorithm 4 builds $\langle \Sigma_k, X_k \rangle$ in $\mathcal{AMP}[\mathcal{L}_k]$.

Proof Proof of the liveness property of Σ_k and the loneliness property of X_k .

Let A be the set of correct processes p_i whose boolean variable $alone_i$ takes the value *true*. Due to line 02, each process p_i of A sets qr_i to $\{i\}$ and, due to lines 05 and 11, it follows that qr_i remains forever equal to $\{i\}$. We consider two cases.

- Case 1: There are k or more faulty processes ($|\mathcal{C}| \leq n - k$).
As $|\mathcal{C}| \leq n - k$, it follows from the loneliness property of \mathcal{L}_k that $A \neq \emptyset$ which establishes the loneliness property of X_k .

It follows from line 03 that each process p_i of A broadcasts forever the message $\text{ALONE}(i)$ and from line 05 that, after all the messages $\text{ALONE}()$ sent by faulty processes have been received, no process p_j will add the identity of a faulty process to its quorum qr_j at line 05. Moreover, after k processes have crashed, the task $T2$ of each correct process p_i not in A eventually remains blocked forever at line 09 and p_i will never add faulty processes to qr_i at line 11. Consequently, there is a time after which no faulty process will ever be added to the quorum qr_i of a correct process from which we conclude that there is a time after which the quorum qr_i of any correct process p_i contains only its identity or its identity plus the identity of another process of A . This concludes the proof of the liveness property of Σ_k when $|\mathcal{C}| \leq n - k$.

Remark. Let us observe that eventually any correct process p_i of A is such that $qr_i = \{i\}$ while any process p_i not in A is such that $qr_i = \{i, j\}$ where p_j is a process of A that can change with time. In the paragraph that follows the proof, the set A is called *kernel*.

- Case 2: There are less than k faulty processes ($|\mathcal{C}| > n - k$).

As $|\mathcal{C}| > n - k$, there are at least $(n - k + 1)$ correct processes that forever broadcast messages `ALIVE()` (line 06) and consequently no process will ever block forever at line 09. Hence, every correct process p_i that does not belong to A will infinitely often updates qr_i to sets of $(n - k + 1)$ correct processes (line 11). Moreover, after it has received the last message `ALONE()` sent by a faulty process, p_i no longer includes a faulty process in its quorum qr_i at line 05. It follows that there is a time after which the quorum of a correct process contains only correct processes which proves the liveness property of Σ_k . The loneliness property of X_k follows trivially from $\neg(|\mathcal{C}| \leq n - k)$.

Proof of the intersection property of Σ_k .

Let us assume by contradiction that there is a set of quorum values $\{q_j\}_{1 \leq j \leq k+1}$ computed by the algorithm that are such that $\forall j_1 \neq j_2 : q_{j_1} \cap q_{j_2} = \emptyset$. Let P be the set of identities of the processes from which these $k + 1$ quorum values have been obtained. As, for any process p_i , we always have $i \in qr_i$ (lines 01, 03, 05 and 11), it follows that $|P| = k + 1$.

Let us assume without loss of generality that $P = \{1, 2, \dots, k + 1\}$ and q_j has been computed by p_j . The value of each q_j has been computed at line 03, 05 or 11 (it cannot be the initial value $qr_j = \Pi$ because that quorum value would intersect any other quorum and our contradiction assumption would not be satisfied).

Let $B \subseteq P$ the set of processes whose quorum values have been computed at line 03 or 05. Hence, each of these quorum values contains a process p_x whose boolean variable $alone_x$ has taken the value *true*. It then follows from the stability property of \mathcal{L}_k that at most k processes p_x can have $alone_x = \text{true}$, from which we conclude that $0 \leq |B| \leq k$. It follows from $B \subseteq P$, $|P| = k + 1$ and $0 \leq |B| \leq k$ that $P \setminus B \neq \emptyset$.

Let $i \in P \setminus B$. Due to the definition of B , p_i has computed q_i at line 11 and, consequently, we have $|q_i| = n - k + 1$. As $\forall j_1, j_2 \in P : (j_1 \neq j_2) \Rightarrow (q_{j_1} \cap q_{j_2} = \emptyset)$ and $\forall i \in P : i \in q_i$, we have $q_i \cap (P \setminus \{i\}) = \emptyset$. It follows that $|q_i| \leq |\Pi| - |P \setminus \{i\}| = n - k$ which contradicts $|q_i| = n - k + 1$ and concludes the proof of the intersection property of Σ_k . $\square_{\text{Lemma 4}}$

A property of $\langle \Sigma_k, X_k \rangle$ The loneliness property of X_k is $(|\mathcal{C}| \leq n - k) \Rightarrow (\exists i \in \mathcal{C}, \exists \tau : qr_i^\tau = \{i\})$. Actually, Algorithm 4 ensures a stronger property, that we call *strong loneliness*, defined as follows.

- Strong loneliness. $(|\mathcal{C}| \leq n - k) \Rightarrow (\exists \tau : \forall \tau' \geq \tau, \forall i \in \mathcal{C} : (\exists j \in qr_i^{\tau'} : \forall \tau'' \geq \tau : qr_j^{\tau''} = \{j\}))$.

When $|\mathcal{C}| \leq n - k$, let us call *kernel* the set of processes p_i that after some time have forever $qr_i = \{i\}$ (this is the set called A in the proof of the liveness property of Σ_k and the loneliness property of X_k in Lemma 4). As it requires that, after some time, each correct process p_i either belongs to the kernel or has a quorum containing a process of the kernel (which can change with time) it is easy to see that strong loneliness property implies loneliness defined by X_k .

As far as the other direction is concerned we have the following. As algorithm 4 ensures strong loneliness, it follows from Lemma 3 and Lemma 4 that this stronger property is implicitly contained in $\langle \Sigma_k, X_k \rangle$.

7 Relating \mathcal{L}_k and Ω_k

This section shows that a simple and pretty natural weakening of the stability of \mathcal{L}_k gives rise to a new failure detector which is equivalent to Ω_k . This establishes a strong relation linking \mathcal{L}_k and Ω_k .

7.1 The failure detector $\diamond \mathcal{L}_k$

Weakening the stability property of \mathcal{L}_k from perpetual to eventual gives rise to the failure detector $\diamond \mathcal{L}_k$. Hence, $\diamond \mathcal{L}_k$ is defined by the following properties.

- Eventual stability. $\exists \tau, \exists K \subset \Pi : (|K| = n - k) \wedge (\forall i \in K, \forall \tau' \geq \tau : alone_i^{\tau'} = \text{false})$.

- Loneliness. $(|\mathcal{C}| \leq n - k) \Rightarrow (\exists i \in \mathcal{C}, \exists \tau : \forall \tau' \geq \tau : \text{alone}_i^{\tau'} = \text{true})$.

Hence, when compared to \mathcal{L}_k , $\Diamond \mathcal{L}_k$ ensures the boolean variables of at least $n - k$ processes remains forever equal to *false* only after an unknown but finite time.

The next two sections show that $\Diamond \mathcal{L}_k$ and Ω_k are equivalent in $\mathcal{AMP}[\emptyset]$, i.e., $\Diamond \mathcal{L}_k$ can be built in $\mathcal{AMP}[\Omega_k]$ and Ω_k can be built in $\mathcal{AMP}[\Diamond \mathcal{L}_k]$. The next theorem follows directly from the lemmas 5 and 7 that are proved in the next two sections.

Theorem 4 $\mathcal{AMP}[\Diamond \mathcal{L}_k]$ and $\mathcal{AMP}[\Omega_k]$ have the same computational power.

7.2 Building $\Diamond \mathcal{L}_k$ in $\mathcal{AMP}[\Omega_k]$

Algorithm 5 is a very simple construction of $\Diamond \mathcal{L}_k$ in $\mathcal{AMP}[\Omega_k]$. The boolean alone_i of each process p_i is initialized to *false*. Then, a process p_i repeatedly updates it to *true* or *false* according to the fact that its identity appears or does not appear in the local output leaders_i currently provided by its underlying failure detector Ω_k .

```

init  $\text{alone}_i \leftarrow \text{false}$ ;
repeat forever  $\text{alone}_i \leftarrow (i \in \text{leaders}_i)$  end repeat.

```

Algorithm 5: Building $\Diamond \mathcal{L}_k$ in $\mathcal{AMP}[\Omega_k]$ (code for p_i)

Lemma 5 Algorithm 5 builds $\Diamond \mathcal{L}_k$ in $\mathcal{AMP}[\Omega_k]$.

Proof Let τ be a finite time after which all faulty processes have crashed and there is a set LD of k process identities such that $LD \cap \mathcal{C} \neq \emptyset, \forall i \in \mathcal{C}, \forall \tau' \geq \tau : \text{leaders}_i^{\tau'} = LD$. (Due to the eventual leadership property of Ω_k , τ and LD do exist.) It follows from the algorithm that, after τ , each process p_i with $i \in LD \cap \mathcal{C}$ executes forever $\text{alone}_i \leftarrow \text{true}$ (Observation O1) while each process p_i with $i \in \mathcal{C} \setminus LD$ executes forever $\text{alone}_i \leftarrow \text{false}$ (Observation O2). Let $\tau' \geq \tau$ be a time instant at which each correct process p_i has updated at least once its boolean variable alone_i .

Proof of the eventual stability property of $\Diamond \mathcal{L}_k$. Let $K = \Pi \setminus LD$ (hence $|K| = n - k$). For the faulty processes of K , let us remember that, by definition, the boolean variable alone_x of a crashed process p_x is equal to *false*. The correct processes p_i of K are such that $i \in \mathcal{C} \setminus LD$ and it follows from observation O2 that the boolean variables of all correct processes in $\mathcal{C} \setminus LD$ remain forever equal to *false* after τ' . Hence, we have $\exists \tau'' \geq \tau', \exists K : (|K| = n - k) \wedge (\forall i \in K, \forall \tau''' \geq \tau'' : \text{alone}_i^{\tau'''} = \text{false})$, which proves the eventual stability property of $\Diamond \mathcal{L}_k$.

Proof of the loneliness property of $\Diamond \mathcal{L}_k$. Let us observe that, due to Ω_k , $\mathcal{C} \cap LD \neq \emptyset$, hence $\exists i \in \mathcal{C} \cap LD$. It then follows from observation O1 that, after τ' , p_i forever executes $\text{alone}_i \leftarrow \text{true}$ which concludes the proof. $\square_{\text{Lemma 5}}$

7.3 Building Ω_k in $\mathcal{AMP}[\Diamond \mathcal{L}_k]$

Underlying principle The idea of the algorithm is the following. We know from $\Diamond \mathcal{L}_k$ that there is a finite time after which there is a set X of at least $n - k$ processes p_i that will no longer have their boolean alone_i equal to *true*. The algorithm strives to capture the complementary set Y of X in order to have Y in the final output LD of Ω_k . Let us observe that we have the following when Y has been captured. If k or more processes are faulty, it follows from the loneliness property of $\Diamond \mathcal{L}_k$ that there is a correct process p_i whose boolean becomes and remains true forever, hence we have $i \in Y$ and we can take any LD such that $Y \subseteq LD$. If less than k processes are faulty, it is relatively easy for the correct processes to agree on any set of k processes (as any such set includes a correct process).

A list of all subsets of size k Let us consider all possible subsets of k processes. Moreover, let us order all of them according to lexicographical ordering when considering each subset as a sorted array. Hence, the set $\{1, 2, \dots, k-1, k\}$ is the first, $\{1, 2, \dots, k-1, k+1\}$ the second, etc., until the set $\{n-k+1, \dots, n-1, n\}$ that is the last one. Let L be this sorted list of all subsets of size k . Moreover, let L' be the list L where $\{1, 2, \dots, k-1, k\}$ is defined as the successor of the last subset $\{n-k+1, \dots, n-1, n\}$.

Finally, let $\text{next_ld}(sbst)$ be the function that returns the subset that follows $sbst$ in L' .

Local variables To attain their goal, the processes execute asynchronous rounds. The local variable r_i contains the current round number of p_i . The current local output computed by p_i to implement Ω_k is kept in the local variable $leaders_i$.

Each process p_i manages a set called $next_set_i$. This set contains all the pairs $(r, leaders)$ received by p_i . The aim of this set is to allow the processes to proceed in the *very same order* from the pair $(1, \{1, 2, \dots, k-1, k\})$, to the pair $(1, \{1, 2, \dots, k-1, k+1\})$, etc., until the pair $(1, \{n-k+1, \dots, n-1, n\})$ during the first round, then from $(2, \{1, 2, \dots, k-1, k\})$ until $(2, \{n-k+1, \dots, n-1, n\})$ during the second round, etc., until they stop during a round r on the very same pair $(r, leaders)$ and define accordingly $LD = leaders$.

```

(01) init  $leaders_i \leftarrow \{1, 2, \dots, k\}; r_i \leftarrow 1; next\_set_i \leftarrow \emptyset;$ 
(02) repeat forever if ( $alone_i$ ) then broadcast  $ALONE(i)$  end if end repeat.

(03) when  $ALONE(j)$  is received: if ( $j \notin leaders_i$ ) then broadcast  $NEXT(r_i, leaders_i)$  end if.

(04) when  $NEXT(r, leaders)$  is received for the first time:
(05)   broadcast  $NEXT(r, leaders);$ 
(06)    $next\_set_i \leftarrow next\_set_i \cup \{(r, leaders)\};$ 
(07)   while  $((r_i, leaders_i) \in next\_set_i)$  do
(08)      $leaders_i \leftarrow next\_ld(leaders_i);$ 
(09)     if ( $leaders_i = \{1, 2, \dots, k\}$ ) then  $r_i \leftarrow r_i + 1$  end if
(10)   end while.

```

Algorithm 6: Building Ω_k in $\mathcal{AMP}[\Diamond \mathcal{L}_k]$ (code for p_i)

Behavior of a process p_i Algorithm 6 describes the behavior of a process p_i . Let us first observe that, if no correct process p_i ever receives a message $ALONE(j)$, we can conclude from the eventual stability property of $\Diamond \mathcal{L}_k$ that less than k processes are faulty. Then no message is ever exchanged among the correct processes p_i and they all agree on $leaders_i = LD = \{1, 2, \dots, k\}$ (that contains at least one correct process) and Ω_k is trivially implemented.

Let us now consider the general case. Each time it reads *true* from $alone_i$, process p_i broadcasts a message $ALONE(i)$. When it receives a message $ALONE(j)$ such that p_j is not currently in $leaders_i$, process p_i broadcasts $NEXT(r_i, leaders_i)$ (line 03). This message is to indicate to the others processes that its current set $leaders_i$ seems not to be the final one and consequently p_i demands the others processes to try the next candidate leader set obtained from the list L' .

When p_i receives a message $NEXT(r, leaders)$ for the first time, it forwards it to all in case the sender crashed during its broadcast (line 05) and saves it in $next_set_i$ (line 06). Then if the current pair $(r_i, leaders_i)$ belongs to $next_set_i$ (line 07), p_i progresses to the next candidate set of the list L' (line 08). If the new value of $leaders_i$ is $\{1, \dots, k\}$, all the elements of L have been tried during round r_i and consequently p_i progresses to the next round (line 09) to try again the elements of L with the aim to stop on one of them.

Let $(r1, leaders1) < (r2, leaders2) \stackrel{def}{=} (r1 < r2) \vee ((r1 = r2) \wedge (leaders1 < leaders2))$ (where $leaders1 < leaders2$ is the order of the sorted list L).

Lemma 6 *Let p_i be a correct process. Let $r_i = r$ and $leaders_i = ld$. Process p_i has received and broadcast all the messages $NEXT(r', ld')$ such that $(r', ld') < (r, ld)$.*

Proof Due to lines 08-09, when modified, the new value of the pair $(r_i, leaders_i)$ is the direct successor pair, according to the order defined on these pairs. It follows from this observation and the initial value of the pair $(r_i, leaders_i)$ (namely, $(1, \{1, \dots, k\})$) that, when $(r_i, leaders_i) = (r, ld)$, the pair $(r_i, leaders_i)$ has taken all the pair values (r', ld') such that $(1, \{1, \dots, k\}) \leq (r', ld') < (r, ld)$.

Let us now observe that the pair $(r_i, leaders_i)$ progresses to its next value only when its current value belongs to the set $next_set_i$ (line 07). Moreover, a pair (r', ld') is added to $next_set_i$ only when a message $NEXT(r', ld')$ is received by p_i for the first time, and this message is then systematically forwarded to all (lines 04-07).

It follows that, when $(r_i, leaders_i) = (r, ld)$, p_i has received and broadcast all the messages $NEXT(r', ld')$ such that $(1, \{1, \dots, k\}) \leq (r', ld') < (r, ld)$. $\square_{\text{Lemma 6}}$

Lemma 7 *Algorithm 6 builds Ω_k in $\mathcal{AMP}[\diamond \mathcal{L}_k]$.*

Proof Let us first observe that the values taken by any set $leaders_i$ are the subsets of size k defined in the list L . The validity property of Ω_k follows trivially.

Let Π_1 denote the set of processes that broadcast an infinity of messages $ALONE()$. It follows from the eventual stability property of $\diamond \mathcal{L}_k$ that there is a finite time τ_1 after which at least $n - k$ boolean variables $alone_i$ remain forever equal to *false*, hence $|\Pi_1| \leq k$. Moreover, there is a time $\tau_2 \geq \tau_1$ from which (a) all received messages $ALONE()$ are from processes of Π_1 and (b) no process broadcasts at line 03 a message $NEXT(r, ld)$ with $\Pi_1 \subset ld$ (this follows from the predicate used in the **if** statement of line 03).

Let $(r0, ld0)$ be the greatest pair such that $\Pi_1 \subseteq ld0$ and there is a message $NEXT(r0, ld0)$ that entailed an update of $leaders_i$ at some correct process p_i (line 08). If there is no such pair, let $(r0, ld0) = (0, \alpha)$ where $next_ld(\alpha) = \{1, \dots, k\}$. Let $(r1, ld1)$ be the smallest pair such that $(r0, ld0) < (r1, ld1)$ and $\Pi_1 \subset ld1$.

It follows from the definition of $(r0, ld0)$ and Lemma 6 that each message $NEXT(r, ld)$ such that $(r, ld) \leq (r0, ld0)$ has been sent by a correct process from which we conclude that each correct process p_i eventually updates $leaders_i$ to $next_ld(ld0)$.

If $\Pi_1 \not\subseteq leaders_i$ at some correct process p_i , it follows from the definition of Π_1 that p_i eventually receives a message $ALONE(j)$ sent by p_j such that $j \in \Pi_1 \setminus leaders_i$. When it receives such a message (line 03), process p_i broadcasts $NEXT(-, leaders_i)$. It follows that, for each pair (r, ld) such that $(r0, ld0) < (r, ld) < (r1, ld1)$, each correct process broadcasts (at line 03 or line 05) a message $NEXT(r, ld)$. Moreover, once all these messages have been received, the correct processes p_i eventually agree on the same set $leaders_i = ld1$ and no longer broadcast $NEXT(-, -)$ messages.

If there are k or more faulty processes, the loneliness property of $\diamond \mathcal{L}_k$ implies that $\Pi_1 \cap \mathcal{C} \neq \emptyset$ and consequently $ld1$ contains at least one correct process. If there are less than k faulty processes, as $|ld1| = k$, $ld1$ contains at least one correct process, which concludes the proof of the lemma. $\square_{\text{Lemma 7}}$

8 Conclusion

This paper has investigated the computability power and explored the relations linking three failure detectors that have been proposed to solve the k -set agreement problem in asynchronous crash-prone message-passing systems. (The k -set agreement problem is a coordination problem that generalizes the consensus problem.) These three failure detectors are the generalized quorum failure detector Σ_k , the generalized loneliness failure detector \mathcal{L}_k and the generalized eventual leader failure detector Ω_k .

The paper has (a) shown that the failure detector Ω_k and the eventual version of \mathcal{L}_k have the same computational power; (b) shown that \mathcal{L}_k is realistic if and only if $k \geq n/2$; and (c) given an exact characterization of the difference between \mathcal{L}_k and Σ_k . Hence, this paper provides us with a better understanding of these failure detectors in the quest for the weakest failure detector to solve the k -set agreement problem in asynchronous message-passing crash-prone systems.

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